Pseudoparticle solutions to the Yang-Mills equation

1986 J. Phys. A: Math. Gen. 19 3693

(http://iopscience.iop.org/0305-4470/19/17/031)

J. Phys. A: Math. Gen. 19 (1986) 3693-3695. Printed in Great Britain

COMMENT

Pseudoparticle solutions to the Yang–Mills equation

Mrityunjoy De, Swapan Konar and Dipankar Ray
Department of Mathematics, Jadavpur University, Calcutta-700032, India

Received 5 February 1986

Abstract. In a recent paper Ma and Xu have obtained a set of equations for the embedding SO(4) pseudoparticle with vanishing energy-momentum tensor in SU(N) Yang-Mills theory and have obtained one particular solution. We completely solve these equations here.

In a recent paper Ma and Xu (1984) have obtained the following equations for SO(4) pseudoparticle solutions with vanishing energy-momentum tensor in SU(N) Yang-Mills theory:

$$u(2\ddot{\phi}_1-e^2\phi_1^3-3e^2\phi_1\phi_2^2)+3(2\dot{\phi}_1+e\phi_1^2+e\phi_2^2)=0 \tag{1a}$$

$$u(2\ddot{\phi}_2-e^2\phi_2^3-3e^2\phi_1^2\phi_2)+6(\dot{\phi}_2+e\phi_1\phi_2)=0. \tag{1b}$$

The gauge field strength $G_{\mu\nu}$ is given by

$$\begin{aligned}G_{\mu\nu}&=\partial_\mu W_\nu-\partial_\nu W_\mu-\mathrm{i}e[W_\mu, W_\nu]\\&=[-2\phi_1+eu(\phi_1^2+\phi_2^2)]I_{\mu\nu}-\phi_2\varepsilon_{\mu\nu\rho\lambda}I_{\rho\lambda}\\&\quad+[2\dot{\phi}_1+e(\phi_1^2+\phi_2^2)](x_\mu I_{\nu\rho}-x_\nu I_{\mu\rho})x_\rho\\&\quad+[\dot{\phi}_2(x_\mu\varepsilon_{\nu\sigma\rho\lambda}-x_\nu\varepsilon_{\mu\sigma\rho\lambda})I_{\sigma\rho}x_\lambda]-2e\phi_1\phi_2\varepsilon_{\mu\nu\sigma\lambda}x_\rho I_{\rho\sigma}x_\lambda\end{aligned}$$

where $I_{\mu\nu}$ are the generators of $\mathscr{D}(R)$ and $\mathscr{D}(R)$ is an N-dimensional representation of the SO(4) group and where the synchrospherically symmetric gauge potential composed of $I_{\mu\nu}$ and x_μ is given by

$$W_\mu(x)=\phi_1(u)I_{\mu\nu}x_\nu+\phi_2(u)\tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}I_{\nu\rho}x_\lambda \qquad u\equiv x^2$$

and a dot indicates the derivative with respect to $u\equiv x^2$.

Ma and Xu obtained the following particular solutions of (1a) and (1b):

$$\phi_1=\frac{1}{e}\left(\frac{1}{u+a^2}+\frac{1}{u+b^2}\right) \tag{2a}$$

$$\phi_2=\frac{1}{e}\left(\frac{1}{u+a^2}-\frac{1}{u+b^2}\right) \tag{2b}$$

where integration constants a^2 and b^2 are chosen greater than 0 so that ϕ_1 and ϕ_2, and hence W_μ, are regular for all X. In the present comment we obtain general solutions of equations (1) as follows.

The set of equations (1) is equivalent to

$$2u(\ddot{\phi}_1+\ddot{\phi}_2)+6(\dot{\phi}_1+\dot{\phi}_2)+3e(\phi_1+\phi_2)^2-e^2u(\phi_1+\phi_2)^3=0 \tag{3a}$$

$$2u(\ddot{\phi}_1-\ddot{\phi}_2)+6(\dot{\phi}_1-\dot{\phi}_2)+3e(\phi_1-\phi_2)^2-e^2u(\phi_1-\phi_2)^3=0. \tag{3b}$$

The above two equations are respectively equivalent to

$$2V_{vv} - 2V + 3eV^2 - e^2V^3 = 0 \tag{4a}$$

$$2S_{vv} - 2S + 3eS^2 - e^2S^3 = 0 \tag{4b}$$

where

$$V = u(\phi_1 + \phi_2)$$

$$S = u(\phi_1 - \phi_2) \tag{5}$$

and

$$v = \ln u.$$

Integration of (4a) and (4b) gives

$$\ln u = \pm 2 \int \frac{\mathrm{d}V}{(e^2V^4 - 4eV^3 + 4V^2 + 4n_1)^{1/2}} + n_1' \tag{6a}$$

and

$$\ln u = \pm 2 \int \frac{\mathrm{d}S}{(e^2S^4 - 4eS^3 + 4S^2 + 4n_2)^{1/2}} + n_2' \tag{6b}$$

where

$$\phi_1 = (V + S)/2u \tag{6c}$$

and

$$\phi_2 = (V - S)/2u. \tag{6d}$$

For $e = 0$ the integration is trivial and for $e \neq 0$ these equations reduce to

$$\ln u = \pm \int \frac{\mathrm{d}Y}{\{Y[(Y-1)^2 + 4n_1e^2]\}^{1/2}} + n_1' \tag{7a}$$

and

$$\ln u = \pm \int \frac{\mathrm{d}Z}{\{Z[(Z-1)^2 + 4n_2e^2]\}^{1/2}} + n_2' \tag{7b}$$

where

$$Y = [eu(\phi_1 + \phi_2) - 1]^2$$

$$Z = [eu(\phi_1 - \phi_2) - 1]^2.$$

In conclusion, therefore, the general solution of equations (1) for $e \neq 0$ is given implicitly by equations (7). For $n_1 = 0$ equation (7a) can be explicitly integrated to give

$$\ln u = \pm \ln \frac{eu(\phi_1 + \phi_2) - 2}{eu(\phi_1 + \phi_2)} + n_1'. \tag{8a}$$

Similarly for $n_2 = 0$, equation (7b) can be explicitly integrated to give

$$\ln u = \pm \ln \frac{eu(\phi_1 - \phi_2) - 2}{eu(\phi_1 - \phi_2)} + n_2'. \tag{8b}$$

If we consider the negative sign of (8a) and (8b) and put the integration constants $n_1' = \ln(-a^2)$ and $n_2' = \ln(-b^2)$, then equations (8a) and (8b) reduce to the particular solution (2a) of equations (1) that have already been obtained by Ma and Xu (1984).

If we consider the positive sign of equations (8*a*) and (8*b*) and set $n_1' = \ln(-1/a^2)$, $n_2' = \ln(-1/b^2)$ then we obtain

$$\phi_1 = \frac{1}{e}\left(\frac{1}{u(1+a^2u)} + \frac{1}{u(1+b^2u)}\right)$$

$$\phi_2 = \frac{1}{e}\left(\frac{1}{u(1+a^2u)} - \frac{1}{u(1+b^2u)}\right).$$

For $n_1 \neq 0$ and $n_2 \neq 0$, the expressions (7*a*) and (7*b*) are in the form of an elliptical integral. These can be integrated numerically by using tables for standard elliptic integrals.

Reference

Ma Z-Q and Xu B-W 1984 *J. Phys. A: Math. Gen.* **17** L389–93